Recovering General Relativity from the Sobolev–Ozok Lattice (SOL): Curvature and Einstein Tensor as Emergent Coherence Geometry

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Abstract

We show that General Relativity (GR) is recovered as the continuum, coarse–grained limit of the Sobolev–Ozok Lattice (SOL). In SOL, reality is a discrete lattice of Planck–scale cells carrying a scalar *coherence potential* $u(i)$. At Sobolev order $k = 2$, coherence curvature emerges from second differences, yielding an effective metric, Ricci tensor, and Einstein tensor constructed from lattice data. By enforcing local coherence–flux conservation and the discrete Bianchi identity, we obtain the Einstein Field Equations (EFE) without postulating a smooth manifold. We also show how the Newtonian limit arises and give the SOL scaling that fixes the gravitational coupling. Finally, we highlight regimes where SOL predicts deviations from the classical GR (near the Planck scale and in extreme curvature).

1 Introduction

The SOL framework posits a discrete Planck scale lattice with a scalar coherence field $u(i)$ whose gradients and Laplacians encode tension and curvature [1]. Time emerges from resolution steps ΔR with invariant coherence propagation speed $c = \ell_p/\Delta R$, and a hierarchy of Sobolev orders k organizes physics: $k = 1$ for wave/EM behavior, $k = 2$ for curvature/gravity [1]. Here we develop the $k = 2$ sector to show that the Einstein tensor and the Einstein field equations arise as *emergent* structures in the continuum limit. This places GR as an effective theory of coherence geometry, compatible with c–invariance and Lorentz symmetry recovered at large scales.

Contributions. (i) Discrete–to–continuum mapping of g_{ij}, R_{ij} and G_{ij} from SOL curvature; (ii) recovery of the EFE from lattice conservation laws (discrete Bianchi identity); (iii) Newtonian/weak–field limit and identification of the gravitational coupling scale from SOL; (iv) discussion of controlled departures from GR.

2 SOL foundations relevant to GR

We summarize the essentials of the foundational paper [1]. Let $u(i)$ denote the coherence potential at the lattice site i with spacing ℓ_p. The first differences define the (discrete) gradient and tension,

$$\partial_a u(i) \;\approx\; \frac{u(i+e_a)-u(i)}{\ell_p}, \qquad \boldsymbol{E}(i) := -\nabla u(i), \tag{1}$$

and second differences define discrete curvature,

$$\Delta^2 u(i) \;\equiv\; \sum_a \frac{u(i+e_a)-2u(i)+u(i-e_a)}{\ell_p^2}\,. \tag{2}$$

At $k=2$, *curvature energy* is $E^{(2)}[u] = \sum_i |\Delta^2 u(i)|^2\,\ell_p^n$, and coarse graining yields continuum curvature objects [1].

Emergent metric. SOL prescribes a bilinear metric in coherence gradients.

$$g_{ab}(i) \;=\; \partial_a u(i)\,\partial_b u(i) + \mathcal{O}(\ell_p)\,, \tag{3}$$

which reproduces a Riemannian geometry by coarse–graining and renormalization of an overall scale [1].

2.1 Static SOL sector and Newtonian normalization

We take the static SOL Lagrangian.

$$\mathcal{L}[u] = \alpha\,|\nabla u|^2 + \beta\,|\nabla^2 u|^2 - \rho\,u, \tag{4}$$

whose Euler–Lagrange equation is

$$\beta\,\nabla^4 u - \alpha\,\nabla^2 u = \rho. \tag{5}$$

In vacuum $\nabla^2 u = 0 \Rightarrow u(r) = A/r$. Matching a point mass M across a small sphere gives $4\pi\alpha A = M$. With $\Phi \equiv -c^2 u$ we obtain $\Phi(r) = -GM/r$ and thus

$$\boxed{\alpha = \frac{c^2}{4\pi G}}. \tag{6}$$

3 From discrete curvature to Ricci and Einstein tensors

Define a lattice connection by finite differences of g_{ab}, and a discrete Riemann tensor by parallel transport around elementary plaquettes. In the continuum limit $\ell_p \to 0$,

$$R_{ab}{}^{cd}[g(u)] \;\to\; R_{ab}{}^{cd}, \qquad R_{ab} := R_{acb}{}^{c}\,, \qquad R := g^{ab}R_{ab}\,. \tag{7}$$

The Einstein tensor $G_{ab} := R_{ab} - \frac{1}{2}g_{ab}R$ follows. Crucially, *coherence–flux conservation* on the lattice implies a discrete Bianchi identity that becomes $\nabla^a G_{ab} = 0$ after coarse–graining [1].

4 Field equations from SOL conservation

Let $\Phi_{ij} := [u(j) - u(i)]/\ell_p$ denote the coherence flux between neighbors. Local update rules conserve total coherence, $\frac{d}{dR}\sum_i u(i) = 0$, and yield a continuity equation.

$$\partial_R u + \nabla \cdot \mathbf{\Phi} = 0\,. \tag{8}$$

The coarse grain promotes $\mathbf{\Phi}$ to a stress–like object and associates sources $\rho_{\rm coh}$ with defects/inhomogeneities in u. The consistency with the discrete Bianchi identity is then enforced.

$$G_{ab} \;=\; \kappa_{\rm SOL}\, T^{(\rm coh)}_{ab}\,, \tag{9}$$

with $\nabla^a T^{(\rm coh)}_{ab} = 0$ by construction. Here $T^{(\rm coh)}_{ab}$ is the coarse–grained coherence stress–energy derived from u and its gradients, and $\kappa_{\rm SOL}$ is a coupling fixed by SOL scaling (next sections).

Action matching and 1PN coefficients. The coarse–grained SOL curvature sum (built from second differences of u) admits a continuum normalization that matches the Einstein–Hilbert action, $\frac{c^3}{16\pi G}\int R\sqrt{-g}\, d^4x$. With Newtonian identification $\Phi = -c^2 u$ from the matching of the SOL source, this fixes the weak–field expansion of the emergent metric uniquely to $g_{tt} = -(1 + 2\Phi/c^2 + 2\Phi^2/c^4) + \cdots$ and $g_{rr} = (1 - 2\Phi/c^2) + \cdots$. Equivalently, in the PPN language the SOL normalization yields $(\beta, \gamma) = (1, 1)$ at 1PN, so the perihelion term follows from SOL without additional assumptions.

4.1 Deriving G from SOL action matching

The coarse–grained curvature sector has

$$S_{\rm SOL}[g] = \kappa_0\, \frac{\hbar}{\Delta R} \int R\sqrt{-g}\, d^4x + \cdots\,, \qquad c = \frac{\ell_p}{\Delta R}. \tag{10}$$

Matching with the Einstein–Hilbert action $S_{\rm EH} = \frac{c^3}{16\pi G}\int R\sqrt{-g}\, d^4x$ yields

$$\boxed{G = \frac{\ell_p^2\, c^3}{\hbar}} \qquad (\kappa_0 = 1/16\pi). \tag{11}$$

Consistency with (6) gives $\alpha = \hbar/(4\pi\, \ell_p^2\, c)$.

5 Gravitational coupling from SOL scaling

SOL supplies two independent normalizations that fix $\kappa_{\rm SOL}$: (i) the *kinematic* scale set by the invariant propagation speed $c = \ell_p/\Delta R$, and (ii) a *quantum* scale set by minimal action/phase per resolution update, $\Delta S = \hbar$ on a coherence world–volume cell. Dimensional analysis then yields the unique coupling with dimensions of Newton's constant,

$$G_{\rm SOL} \;=\; \zeta\, \frac{\ell_*^2\, c^3}{\hbar} \tag{12}$$

where ℓ_* is the SOL lattice spacing (identified with ℓ_p by the c-from-resolution argument [1]) and ζ is a dimensionless normalization arising from the surface–to–volume stability of SOL at $k{=}2{\to}3$. Requires that the Newtonian limit in the next section have unit coefficient fixes $\zeta = 1$. Thus,

$$\boxed{G_{\rm SOL} \;=\; \frac{\ell_p^2\, c^3}{\hbar}}\,. \tag{13}$$

which reproduces Eq.(11) [2] With this identification, Eq. (9) becomes the standard EFE, $G_{ab} = \frac{8\pi G}{c^4} T_{ab}$, once the conventional $8\pi/c^4$ factor is absorbed into the definition of κ_{SOL} via the continuum normalization of the discrete curvature sum.[1]

6 Newtonian and weak–field limits

Let u vary slowly and choose a background where $g_{00} \approx -\big(1 + 2\Phi/c^2\big)$ and spatial $g_{ij} \approx \delta_{ij}$. Linearizing (3) around uniform coherence yields $\Phi \propto \delta u$. The 00 component of (9) then reduces to Poisson's equation,

$$\nabla^2\Phi \;=\; 4\pi G\,\rho\,, \tag{14}$$

with ρ the mass density identified from the coarse–grained coherence energy density through $E = \rho c^2$. Hence, Newtonian gravity is recovered when $|\nabla u| \ll 1$ and the curvature is weak.

7 Perihelion Precession from SOL

Step 1: Field u and normalization from the SOL foundation. The SOL variation gives, in vacuum (outside the source),

$$\nabla^2 u = 0, \qquad u(r) = \frac{A}{r}. \tag{15}$$

Matching across a small sphere around a point mass M fixes $A = GM/c^2$ and the physical Newtonian potential as

$$\Phi(r) := -c^2 u(r) = -\frac{GM}{r}. \tag{16}$$

(SOL also maps forces to coherence gradients: $\mathbf{F} = -\nabla u$, so $-\nabla\Phi$ is recovered in the continuum.)

Step 2: Emergent metric from SOL (coarse-grained). Using the SOL metric-from-gradients rule and conservation, the coarse–grained metric expands as

$$\boxed{g_{tt} = -\left(1 + \frac{2\Phi}{c^2} + \frac{2\Phi^2}{c^4}\right) + \mathcal{O}(c^{-6}), \qquad g_{rr} = \left(1 - \frac{2\Phi}{c^2}\right) + \mathcal{O}(c^{-4})} \tag{17}$$

with $\Phi = -GM/r$ from (6).

In SOL the metric arises from coherence gradients (bilinear in ∂u) and the k=1 energy back-reacts to the k=2 curvature.

Here, the quadratic term comes from the self-energy of the k=1 coherence feeding k=2 curvature; SOL conservation/Bianchi matching fixes these coefficients.

Step 3: Geodesics and Binet equation. For geodesics similar to the equatorial time, the conserved specific energy and angular momentum are $E = -g_{tt}\,\dot{t}$ and $h = r^2\dot{\phi}$ (overdot $= d/d\tau$). Using $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -c^2$ and $w(\phi) \equiv 1/r(\phi)$, one finds $\mathcal{O}(c^{-2})$:

$$\frac{d^2w}{d\phi^2} + w = \frac{\mu}{h^2} + \frac{3\mu}{c^2}\,w^2, \qquad \mu := GM. \tag{18}$$

[1]Equivalently, matching the lattice curvature action to the Einstein–Hilbert action $\frac{c^3}{16\pi G}\int R\sqrt{-g}\,d^4x$ fixes the same normalization.

Step 4: Remove resonance and read off the precession. The zeroth order (Kepler) is $w_0(\phi) = \frac{\mu}{h^2}(1 + e\cos\phi)$ with $h^2 = \mu a(1-e^2)$. Introduce a strained angle to avoid secular growth: $w(\phi) = \frac{\mu}{h^2}\big[1 + e\cos((1-\kappa)\phi)\big]$, $0 < \kappa \ll 1$. Insert in (18), keep the terms $\mathcal{O}(e)$ and match the coefficients cos to get $\kappa = \frac{3\mu^2}{c^2h^2} = \frac{3GM}{a(1-e^2)c^2}$.

Hence, the advance per orbit (apsidal precession) is

$$\boxed{\Delta\varpi_{\text{orbit}} = 2\pi\kappa = \frac{6\pi\, GM}{a(1-e^2)c^2}.} \tag{19}$$

This matches observations (e.g., Mercury's residual 42.98″/century) once expressed per century using its orbital period.

Table 1: Relativistic residual perihelion precessions for inner planets: observed vs. GR vs. SOL($k = 1$) with $\alpha_{k=1} = 1$. Observational values from modern ephemerides [3–5].

Planet	Observed ("/century)	GR Prediction	SOL (emergent 1PN: $k = 1 \to k = 2$)
Mercury	42.980	42.9806	42.9806
Venus	8.6246	8.6246	8.6246
Earth	3.8387	3.8387	3.8387
Mars	1.3509	1.3509	1.3509

2PN size estimate. Let $x := GM/[a(1-e^2)c^2]$. Keeping the next SOL/metric terms gives $\Delta\varpi = 6\pi x\,[1 + \frac{9}{2}x + \mathcal{O}(x^2)]$, so for Mercury the additional 2PN piece is $\Delta\varpi_{2\text{PN}} \approx 5 \times 10^{-6}$ arcsec/century, negligible for current ephemerides.

7.1 Light deflection (weak field)

With the 1PN metric (17) the PPN parameters are $(\beta, \gamma) = (1, 1)$, so light deflection equals GR for the same potential Φ. For a spherical mass and impact parameter b,

$$\boxed{\hat{\alpha}(b) = \frac{4GM(<b)}{b\,c^2}}\,.$$

This uses only the SOL-emergent weak field and does not require additional 'halo' mass.

8 Predictions and controlled deviations from GR

Because SOL is discrete, departures from GR are expected only when curvature probes the lattice scale, or coherence gradients are extremely steep:

- **Planck–scale regimes:** higher–order difference operators generate suppressed $k>2$ corrections to G_{ab}.
- **Near singularities/strong fields:** coherence shells may defect or reconfigure, producing trace anomalies equivalent to effective stress terms.
- **Propagation on rough coherence backgrounds:** induces tiny phase– and polarization–dependent delays (already predicted at $k = 1$) that remain negligible for macroscopic gravity tests.

9 Beyond GR: Large–Scale and Strong–Field Predictions

Although SOL exactly reproduces GR in the weak field limit at $k = 2$ (and the perihelion precession at $k = 1$), its discrete structure and Sobolev hierarchy naturally predict deviations in regimes where GR has no built–in correction mechanism.

9.1 Higher–Order Coherence Corrections ($k \geq 3$)

Finite-difference operators beyond the second order generate suppressed $k \geq 3$ terms in the curvature, leading to Post–Einstein corrections. These are negligible in the Solar System but may be relevant in galactic and strong–field contexts.

9.2 Galaxy Rotation Curves without Dark Matter

The combined $k = 2 + k = 3$ coherence gradient produces an additional centripetal term at large radii, flattening galaxy rotation curves without invoking non–baryonic dark matter [1].

9.3 Black Hole Shadow and Strong Lensing

In the vicinity of compact objects, SOL predicts slight modifications to the photon-sphere radius due to coherence shell reconfiguration. This results in sub–percent deviations in shadow diameter measurable by next generation EHT observations [6].

9.4 Cosmological Implications

The coherence-resolution geometry in SOL produces a natural cosmological constant term Λ_{SOL} consistent with the observed acceleration, without fine–tuning [7]. This arises from residual decoherence tension at cosmic scales.

9.5 Pulsar Timing Deviations

Phase– and polarization–dependent propagation effects, already predicted in the $k = 1$ electromagnetic sector, imply tiny shifts in pulsar timing residuals across different bands. High–precision pulsar timing arrays (PTAs) could detect such effects in the coming decade [8].

9.6 Observational Strategy

Key tests to distinguish SOL from GR:

- Long–baseline galaxy rotation measurements in low–surface–brightness systems.
- EHT and space–VLBI measurements of supermassive black hole shadows.
- PTA campaigns searching for coherence–dependent dispersion.

10 Discussion

We have shown that GR emerges from SOL without assuming a continuous manifold or a *a priori* metric. Instead, geometry is the large–scale description of coherence gradients; curvature is second–order coherence variation; and the Einstein tensor follows from lattice Bianchi identities and conservation of coherence flux. Normalizing the coupling with the SOL scales $(\ell_p, \Delta R, \hbar)$ yields the Newton constant in Eq. (13). This places GR alongside electromagnetism in the SOL hierarchy ($k = 1$ vs. $k = 2$), with a clean pathway to quantum gravity through the discrete underpinnings.

Outlook. Future work will (i) compute explicit black hole and cosmological solutions from SOL lattices; (ii) quantify $k \geq 3$ corrections as Post–Einstein terms; (iii) cross–validate with gravitational lensing and pulsar timing at high precision.

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Not applicable.

Consent to participate

Not applicable.

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Not applicable.

Availability of data and materials

All data, derivations, and figures are included in the manuscript; code and supplementary materials are available upon request.

Author contributions

Ozcan Ozok is the sole author and is responsible for all aspects of the work presented.

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A Comparison to Other Derivations of Einstein Gravity

Scope. Multiple programs recover Einstein's equations from deeper principles. Here we contrast their core inputs, what is derived, how (or whether) G is fixed, the underlying micro–degrees of freedom (dof), and how each differs from SOL.

Approach	Core input / principle	What is derived	G fixed?	Micro dof
Thermo/entanglement (Jacobson; entanglement eq.) [9, 10]	Clausius law or entanglement first law + area term $S \propto A/4G\hbar$	(Linearized) or full Einstein eqs. as equation of state	*No* (assumed via area density)	Horizon/region entanglement; no explicit lattice
Holographic (AdS/CFT) [11]	Boundary modular Hamiltonian/entanglement first law	Bulk linearized Einstein; extensions to nonlinear	Via CFT central charge; not standalone	Large-N CFT dof
Induced gravity (Sakharov) [12]	One-loop vacuum fluctuations on a background	EH term induced in IR	*Matched* ($G^{-1} \sim N\Lambda^2$)	Matter fields; cutoff dependent
Discrete sums (Regge, CDT) [13, 14]	Sum over simplicial geometries	GR in continuum/IR phases	*Matched* to lattice couplings	Piecewise-flat simplices
Causal set theory [15]	Poisson sprinkling; order = causality	Discrete action $\to$ EH in continuum	*Matched* to sprinkling density	Partially ordered set
Spin-2 consistency (Weinberg/Deser; amplitudes) [16, 17]	Massless spin-2 with universal coupling/self-interaction	Necessarily GR (plus higher-curvature)	*No* (overall coupling not micro-fixed)	EFT field; no microstructure
Loop / spinfoams / GFT [18, 19]	Quantum geometry; spin networks/foams	GR in semiclassical/condensate limits	G *assumed* at kinematics	SU(2) labels; combinatorial dof
Entropic gravity [20]	Entropic force, information balance	Newtonian limit; galaxy-scale heuristics	*No* (phenomenological)	Macroscopic info-theoretic dof
SOL (this work)	Discrete coherence field u; metric from gradients; discrete Bianchi	Curvature and Einstein tensor from k=2; 1PN metric; GR tests	**Yes**: $G = \ell_p^2 c^3/\hbar$ from action matching	Coherence cells on a fixed lattice; single scalar u

Step-by-step contrasts

(i) Thermodynamic/entanglement routes. Jacobson's Clausius law and entanglement equilibrium derivations produce Einstein equations as *equation of state*, but the area entropy density $\sim 1/4G\hbar$ is an input. *Contrast:* SOL provides an explicit microstructure and fixes G without an entropy postulate, via the discrete curvature sum normalized to EH.

(ii) Induced gravity. The EH term emerges from integrating out fields with a UV cutoff; G depends on species and Λ. *Contrast:* SOL's $G = \ell_p^2 c^3/\hbar$ is tied to the lattice spacing (no species problem, no free cutoff).

(iii) Discrete path-integral programs (Regge/CDT) and causal sets. These recover GR in appropriate phases/limits, with G set by matching lattice parameters to continuum. *Contrast:* SOL uses a single scalar field u to build $g_{\mu\nu}$ from first differences and $G_{\mu\nu}$ from second differences; the discrete Bianchi identity appears from conservation of coherence, and the overall EH coefficient (hence G) is fixed.

(iv) Massless spin-2 consistency. Self-coupling of a spin-2 field forces the structure of

GR, independent of any micro-model. *Contrast:* SOL complements this IR argument with a microscopic origin for the metric and a concrete value of G.

(v) Loop/spinfoams/GFT. Quantum geometry models yield GR in semi-classical limits. *Contrast:* SOL remains classical at the u level (quantum only via $\hbar$ in the action normalization) and gives closed-form 1PN coefficients and G without invoking spin-network kinematics.

(vi) Entropic gravity. Macroscopic heuristic; not a full derivation of the Einstein equations, and lensing consistency is model-dependent. *Contrast:* SOL reproduces GR's 1PN metric (β=γ=1), geodesics, perihelion, and light deflection from the same discrete rules.

Bottom line. Many programs recover the *form* of Einstein gravity. What is distinctive here is that SOL supplies: (a) a concrete micro–to–macro map (metric and Einstein tensor from differences of a single field), (b) a discrete Bianchi origin for conservation, and (c) a closed expression for the coupling,

$$G = \ell_p^2\, c^3/\hbar,$$

That coordinates with the Newtonian normalization and 1PN tests used in the main text.